ENDOMORPHISM RINGS OF ESSENTIAL EXTENSIONS OF A NOETHERIAN MODULE

BY

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ABSTRACT

Nil subrings of the ring of endomorphisms of the rational completion of a noetherian module are nilpotent. If the quasi-injective hull of a noetherian module is contained in its rational completion, then the ring of endomorphisms of the former is semi-primary.

In this paper we investigate the following question. If N_R is an essential extension of a noetherian module M_R, and if P is the ideal of endomorphisms of N with large kernels, does some power of P annihilate M? We show this to be the case provided there exists an integer k such that P^kM is contained in $\bar{M}$, the rational completion of M. Therefore, when M is quasi-injective or when the rational completion of M contains its quasi-injective hull, then for J, the Jacobson radical of the endomorphism ring of the injective hull of M, we have $J^nM = 0$ for some positive integer n. The procedure used here is similar to that of Fisher [1], [2]. Our results extend those of Goldie and Small [3], Shock [5], and the aforementioned papers of Fisher.

All rings considered in this paper have an identity and all modules are unital. We adopt the convention of writing mappings on the opposite side of the scalars.

In what follows, N_R denotes an essential extension of a module M_R, S denotes any subring of the ring of endomorphisms of N and $P = \{\alpha \in S\colon \ker \alpha \text{ is a large submodule of } N\}$. An ideal A of S is termed $M - T$ nilpotent if for any sequence $\{\phi_i\}$ of elements of A, a positive integer n can be found such that $\phi_n \cdots \phi_1 M = 0$. Analogous to the notion of annihilators, we define M-annihilators of subsets of S. For a subset A of S, $(A^r\colon M)$ and $(A^l : M)$ are respectively defined by

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$\{\alpha \in S: A\alpha M = 0\}$ and $\{\alpha \in S: \alpha AM = 0\}$. In general when M and S are fixed, we shall denote the above by A^r and A respectively. A^r is an additive subgroup and A^l is a left ideal of S. These annihilators have usual properties such as $A \subseteq A^{lr}$, $A \subseteq A^{rl}$, $A^l = A^{lrl}$ and $A^r = A^{rlr}$. For a fixed M, the $a \cdot c \cdot c \cdot (d \cdot c \cdot c \cdot)$ on left M-annihilators in S is equivalent to $d \cdot c \cdot c \cdot (a \cdot c \cdot c \cdot)$ on right M-annihilators.

THEOREM 1. *If M_R is noetherian, then P is $M-T$ nilpotent.*

PROOF. Let $\{\phi_i\}$ be a sequence of elements of P. Define $s_n = \phi_n \cdots \phi_1$ for $n = 1, 2, \cdots$. Consider the chain $\ker s_1 \cap M \subseteq \ker s_2 \cap M \cdots$ of submodules of M. For some integer k, we must have $\ker s_k \cap M = \ker s_{k+1} \cap M = \cdots$. If x is any element of $s_k M \cap \ker \phi_{k+1} \cap M$, then $x = s_k y$ for some y in M, and $0 = \phi_{k+1} x = s_{k+1} y$ implies $s_k y = x = 0$. Thus $s_k M \cap \ker \phi_{k+1} \cap M = 0$, and since M and $\ker \phi_{k+1}$ are both large in N, we have $\phi_k \cdots \phi_1 M = 0$.

In particular, by choosing $\phi_i = \phi$, a fixed element in P, for $i = 1, 2, \cdots$, we have the following corollary.

COROLLARY 2. *If $\phi \in P$, then for some positive integer n, $\phi^n M = 0$.*

THEOREM 3. *If M is noetherian and S satisfies $a \cdot c \cdot c \cdot$ on right M- annihilators, then there exists an integer n such that $P^n M = 0$.*

PROOF. Suppose if possible, $P^n M \neq 0$ for $n = 1, 2, \cdots$. From the chain of right M-annihilators, $P^r \subseteq (P^2)^r \subseteq \cdots$, we have, for some integer k, $(P^k)^r = (P^{k+1})^r = \cdots$. Since $P^{k+1} M \neq 0$, there exists $\alpha_1 \in P$ such that $P^k \alpha_1 M \neq 0$. Then $\alpha_1 \notin (P^k)^r = (P^{k+1})^r = \cdots$ and consequently $P^n \alpha_1 M \neq 0$ for each $n = 1, 2, \cdots$. In particular, $P^{k+1} \alpha_1 M \neq 0$ implies the existence of $\alpha_2 \in P$ such that $P^k \alpha_2 \alpha_1 M \neq 0$. Continuing in this manner, we obtain a sequence $\{\alpha_n\}$ of elements of P for which $\alpha_n \cdots \alpha_1 M \neq 0$ for each n. This is in contradiction with Corollary 2.

THEOREM 4. *Let M_R be noetherian. Let N be any essential extension and $\bar{M}$ the rational completion of M in the injective hull $E(M)$ of M. If $SM \subseteq \bar{M} \cap N$, then $P^n M = 0$ for some positive integer n.*

PROOF. We will show that S satisfies $a \cdot c \cdot c \cdot$ on right M-annihilators and then the result follows from Theorem 3. Let $A_1 \subseteq A_2 \subseteq \cdots$ be a chain of right M-annihilators in S. From the chain $A_1 M \cap M \subseteq A_2 M \cap M \cdots$ of submodules of M, we find an integer t such that $A_t M \cap M = A_{t+1} M \cap M = \cdots$. If $\phi \in A_t^l$, then $\phi A_t M = 0$ which implies $\phi(A_t M \cap M) = \phi(A_{t+1} M \cap M) = 0$. We show $\phi A_{t+1} M \cap M = 0$. If $m \in \phi A_{t+1} M \cap M$, then $m = \phi x$ for some $x \in A_{t+1} M \subseteq \bar{M}$.

Then $x(x^{-1}M) \subseteq A_{t+1}M \cap M$ where $x^{-1}M = \{y \in R: xy \in M\}$. Thus $\phi x(x^{-1}M) \subseteq \phi(A_{t+1}M \cap M) = 0$ which implies $\phi x = m = 0$ by [6; Lem. 1.1, p. 621]. Thus $A_t^l \subseteq A_{t+1}^l$ and we have $A_{t+1} = A_{t+1}^{lr} \subseteq A_t^{lr} = A_t$.

As a consequence of Theorem 4, we obtain a number of interesting corollaries.

If for some integer k, we have $P^kM \subseteq \bar{M} \cap N$ then by choosing $P^k = S$ we have the following corollary.

COROLLARY 5. *If $P^kM \subseteq \bar{M} \cap N$ for some integer k, then $P^kM = 0$ for some integer n.*

Next, if we choose N to be the injective hull of M and $S = \text{Hom}(N, N)$, P coincides with the Jacobson radical of S. Further, the condition of Theorem 4 will be satisfied if either M is quasi-injective or if its quasi-injective hull is contained in $\bar{M}$. Thus we have Corollary 6.

COROLLARY 6. *Let N be the injective hull of a noetherian module M and P the Jacobson radical of* Hom(N, N). *If M is quasi-injective or if the quasi-injective hull of M is contained in its rational completion, then $P^nM = 0$ for some positive integer n.*

Suppose N is the rational completion of M. Then we have $P^nM = 0$, which implies that $P^nN = 0$ because N is a rational extension of M [4; Ex. 5, p. 104]. Thus P is nilpotent. We can now apply a result of Shock [5, Th. 3] and conclude that nil subrings of S are nilpotent.

COROLLARY 7. *If N is the rational completion of M, then P is nilpotent and nil subrings of S are nilpotent.*

Finally suppose $S = \text{Hom}(N, N)$, where N is the quasi-injective hull of M contained in the injective hull $E(M)$ of M. If K is the ring of endomorphisms of $E(M)$ and J its Jacobson radical, as in [5], K/J is isomorphic with S/P and hence S/P is semiperfect. Further if $N \subseteq \bar{M}$, then $P^nM = 0$ implies $P^nN = 0$ and consequently S is semiprimary.

COROLLARY 8. *The ring of endomorphisms of the quasi-injective hull N of a noetherian module M is semiprimary provided $N \subseteq \bar{M}$.*

We give an example illustrating some of the above results where M is quasi-injective.

EXAMPLE. Consider for a fixed prime p, $I = Z_p^\infty$ as a Z-module. Then I_Z is an injective artinian module which is not noetherian and its endomorphism ring consists of the p-adic integers. For a fixed positive integer k, consider the submodule $M = \{a/p^m : a \in Z,\ m \leqq k,\ 0 \leqq a < p^m,\ (a,p) = 1\}$ of I. M is trivially noetherian and it is quasi-injective as well as rationally complete.

The ideal P of endomorphisms wit hlarge kernels in $\mathrm{Hom}(I,I)$ is not nilpotent (in fact, $\mathrm{Hom}(I,I)$ is an integral domain) yet P^k annihilates M. This also shows that, in general, the ring of endomorphisms of the injective hull of a noetherian module is not semiprimary.

References

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